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Is there a Wong-Zakai approximation for big order generators?

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1 Introduction

Let us consider a compact Riemannian manifold M of dimension d endowed with its normalized Riemannian measure dx $(x \in M)$.

Let us consider m smooth vector fields X_i (We will suppose later that they are withous divergence). We consider the second order differential operator:

$$L = 1/2 \sum_{i=1}^{m} X_i^2 \tag{1}$$

It generates a Markovian semi-group P_t which acts on continuous function f on M

$$\frac{\partial}{\partial t} P_t f = L P_t f \; ; P_0 f = f \tag{2}$$

$P_t f \geq 0$ if $f \geq 0$. It is represented by a stochastic differential equation in Stratonovitch sense ([3])

$$P_t f(x) = E[f(x_t(x))] \tag{3}$$

where

$$dx_t(x) = \sum_{i=1}^{m} X_i i(x_t(x)) dw_t^i \; ; x_0(x) = x \tag{4}$$

where $t \to w_t^i$ is a flat Brownian motion on $\mathbb{R}^m$ Classically, the Stratonovitch diffusion $x_t(x)$ can be approximated by its Wong-Zakai approximation.

Let $w_t^{n,i}$ be the polygonal approximation of the Brownian path $t \to w_t^n$ for a subdivision of $[0, 1]$ of length n.

We introduce the random ordinary differential equation

$$dx_t^n(x) = \sum_{i=1}^m X_i(x_t^n(x))dw_t^{n,i} \ ; x_0^n(x) = x \tag{5}$$

Wong-Zakai theorem ([3]) states that if f is continuous

$$E[f(x_t^n(x))] \to E[f(x_t(x))] \tag{6}$$

We are motivated in this paper by an extension of (6) to bigger order generators.

Let us consider the generator $L^k = (-1)^k \sum_I^m X_i^{2k}$. We suppose that the vector fields X_i spann the tangent space of M in all point of M and that they are divergent free. L^k is an elliptic postive essentially self-adjoint operator [1] which generates a contraction semi-group P_t^k on $L^2(dx)$

Let $L^{f,k}$ be the generator on $\mathbb{R}^m$ ($(w_i) \in \mathbb{R}^m$) $\sum(-1)^k \frac{\partial^{2k}}{\partial w_i^2 k}$. By [1], it generates a semi-group $P_t^{f,k}$ on $C(\mathbb{R}^m)$, the space of continuous functions on the flat space endowed with the uniform topology, which is represented by an heat-kernel:

$$P_t^{f,k}[f](w_0) = \int_{\mathbb{R}^m} f(w + w_0) p_t^{f,k}(w) \otimes dw_i \tag{7}$$

($w = (w_i)$) In [7], we noticed that heuristically $P_t^{f,k}$ is represented by a formal path space measure $Q^{f,k}$ such that

$$\int_E f(w_t^k + w_0) dQ^{f,k}(w_.) = P_t^{k,f}(f)(w_0) \tag{8}$$

If we were able to construct a differential equation in the Stratonovitch sense

m

$$dx_t^k(x) = \sum_{i=1}^{m} X_i(x_t^k(x))dw_{t,i}^k \; ; x_0^k(x) = x \tag{9}$$

$$P_t^{f,k}(x) = \int f(x_t^k(x))dQ^{f,k} \tag{10}$$

These are formal considerations because in such a case the path measures are not defined. We will give an approach to (11) by showing that some convenient Wong-Zakai approximation converge to the semi-group. We introduce, according to [4] and [5] the Wong-Zakai operator

$$Q_t^k[f](x) = \int_{\mathbb{R}^m} f(x_i(w)(x))p_t^{f,k}(w) \otimes dw_i = \int_{\mathbb{R}^m} f(x(t^{1/2k}w)(x))p_1^{f,k}(w)dw \tag{11}$$

where

$$dx_1(w)(x) = \sum_{i=1}^{m} X_i(x_s(w))w_i ds \; ; x_0(w)(x) = x \tag{12}$$

As a first theorem, we state:

Theorem 1 *(Wong-Zakai)Let us suppose that the vector fields X_i commute. Then $(Q_{1/n}^k)^n(f)$ converge in $L^2(dx)$ to $P_1^k f$ if f is in $L^2(dx)$*

To give another example, we suppose that M is a compact Lie group G endowed with its normalized Haar measure dg and that the vector fields X_i are elements of the Lie algebra of G considered as right invariant vector fields. This means ,that if we consider the right action on $L^2(dg)$ R_{g_0}

$$f \to (g \to (f(gg_0)) \tag{13}$$

we have

$$R_{g_0}[X_i f](.) = X_i[R_{g_0} f](.) \tag{14}$$

We consider the rightinvariant elliptic differential operator

$$L^k = (-1)^k \sum_{i=1}^{m} X_i^{2k} \tag{15}$$

It is an elliptic positive essentially selfadjoint operator. By elliptic theory ([1]), it has a positive spectrum λ associated to eigenvectors f_λ. $\lambda \geq 0$ if λ belongs to the spectrum.

Theorem 2 *(Wong-Zakai) Let $f = \sum a_\lambda f_\lambda$ such that $\sum_\lambda |a_\lambda|^2 C^\lambda < \infty$ for all $C > 0$. Then $(Q_{1/n}^k)^n(f)$ converges in $L^2(dg)$ to $P_1^k f$.*

We refer to the reviews [5], [6] [7] for the study of stochastic analysis without probability. See [8] for an expended version of this work.

2 Proof of theorem 1

L^k is an elliptic positive operator. By elliptic theory [?], it has a discete spectrum λ associated to normalized eigenfunctions f_λ. Since $\int_{\mathbb{R}^m} |p_1^{f,k}(w)|^2 dw < \infty$, $Q^k_{1/n}$ is a bounded operator on $L^2(dx)$. Moreover

$$Q^k_{1/n} f = \sum a_\lambda Q^k_{1/n} f_\lambda \tag{16}$$

if

$$f = \sum a_\lambda f_\lambda \tag{17}$$

The main remark is that we can compute explicitely $Q^k_{1/n} f_\lambda$. We put $t = 1/n$. Formally

$$f_\lambda(x(t^{1/2k} w))(x) = \sum_{n'} 1/n'!(\sum_m X_\imath w_\imath t^{1/2k}))^{n'}(f_\lambda)(x) \tag{18}$$

Namely, by ellipticity and because the vector fields $X_\imath$ commute with L^k, we can conclude that the L^2-norm of $X_{\imath_1}^{\alpha_1} X_{\imath_2}^{\alpha_2} ... X_{\imath_l}^{\alpha_l} f_\lambda$ has a bound in $\lambda^{\sum \alpha_\imath /2k} C^{\sum \alpha_\imath}$ in order to deduce that the series in (18) converges. It is enough to compute

$$1/n'! \int_{\mathbb{R}^m} (\sum X_\imath w_\imath t^{1/2k}))^{n'} f_\lambda(x) p_1^{k,f}(w) dw = B_{n'} \tag{19}$$

The main remark is if one of the l_i is not a multiple of $2k$, we have

$$\int_{\mathbb{R}^m} w_1^{l_1}..w_m^{l_m} p_1^{k,f}(w)dw = 0 \tag{20}$$

On the other hand, by using the semi-group properties of $P_t^{k,f}$, we have

$$\int_{\mathbb{R}^m} w_1^{2kl_1}..w_m^{2kl_m} p_1^{k,f}(w)dw = \frac{(2kl_1)!}{l_1!}...\frac{(2kl_m)!}{l_m!} \tag{21}$$

Therefore, $B_{n'} = 0$ if n' is not a multiple of $2k$ and is equal because the vector field commute, ,if $n' = 2kl'$ to

$$\frac{1}{(2kl)'!}\sum X_1^{2kl'1}..X_m^{2kl'_m}\frac{(2kl_1)!}{l_1!}...\frac{(2kl_m)!}{l_m!} \frac{(2kl')!}{(2kl'_1)!..(2kl'_m)!} f_\lambda = 1/l'!(L^k)^{l'} f_\lambda \tag{22}$$

We deduce that

$$Q_{1/n}^k f_\lambda = \exp[-1/n\lambda] f_\lambda \tag{23}$$

and that

$$(Q_{1/n}^k)^n f_\lambda = \exp[-\lambda] f_\lambda \tag{24}$$

such that

$$(Q_{1/n}^k)^n f = \exp[-L] f \tag{25}$$

if $f = \sum a_\lambda f_\lambda$.

3 Proof of Theorem 2

Let E_λ be the space of eigenfunctions associated to the eigenvalue λ of L^k. Since L^k commute with the right action of G, E_λ is a representation for the right action of G ([2]). Therefore rightinvariant vector fields acts on E_λ. If Z is a rightinvariant vector field, we can consider the L^2 norm of Zf_λ for f_λ belonging to E_λ. We remark that $(L^k + C)^{1/2k}$ is an elliptic pseudodifferential operator of order 1 (C is strictly ppositive). By Garding inequality [1],

$$\|Zf_\lambda\|_{L^2(G)} \leq C\|f_\lambda\|_{L^2(G)} + \|(L^k + C)^{1/2k} f_\lambda\|_{L^2(G)} \tag{26}$$

f_λ is an eigenfunction associated to $(L^k + C)^{1/2k}$ and the eigenvalue $(\lambda + C)^{1/2k}$.

Let us consider a polynomial $X_{i_1}^{\alpha_1}...X_{i_l}^{\alpha_l} = Z_l$. It acts on E_λ and is norm is bounded by $((\lambda + C)^{1/2k} + C)^{\sum \alpha_i}$ for the L^2 norm.

From that we deduce that if f_λ is an eigenfunction associated to λ of L^k that the series

$$\sum_l \frac{(X_i t^{1/2k} w_i)^l}{l!} f_\lambda \tag{27}$$

converges and is equal to $f_\lambda(x(t^{1/2k}w)(x))$ By distinguishing if w is big or not and using (20), we see that if $l \neq 2kl'$

$$\int_{\mathbb{R}^m} (\sum_{i=1}^{m} X_i t^{1/2k} w_i)^{l'} f_\lambda p_1^{f,k}(x)dw = 0 \tag{28}$$

Moreover, by (20) and (21)

$$\frac{1}{(2kl')!}\int_{\mathbb{R}^m} (\sum_{i=1}^{m} X_i t^{1/2k} w_i)^{2kl'} f_\lambda p_1^{f,k}(x)dw =$$
$$\frac{t^{l'}}{(2kl')!}\int_{\mathbb{R}^m} \sum X_{\alpha_1}..X_{\alpha_{2kl'}} f_\lambda w_1^{2kl'_1}..w_m^{2kl'_m} p_1^{f,k}(w)dw \tag{29}$$

where $2kl'_j$ is the number of of α_i equal to j. By using (20) and (21), we recognize in (29)

$$\frac{1t^{l'}}{(2kl')!}\sum_{\alpha_i} X_{\alpha_1}..X_{\alpha_{2kl'}} f_\lambda \frac{(2kl'_1)!}{l'_1!}...\frac{(2kl'_m)!}{l'_m!} \tag{30}$$

For $l' = 1$, we recognize tL. Let us compute the L^2 norm of the previous element. It is bounded by

$$\frac{t^{l'}}{(2kl')!}\sum_{\alpha_i}(\lambda^{1/2k}+C)...(\lambda^{1/2k}+C)\frac{(2kl'_1)!}{l'_1!}...\frac{(2kl'_m)!}{l'_m!} \tag{31}$$

For $l' = 1$, we recognize tL.

We recognize in the previous sum

$$\frac{t^{l'}}{(2kl')!}\sum \frac{(2kl')!}{(2kl'_1)!...(2kl'_m)!)}\frac{(2kl'_1)!}{l'_1!}...\frac{(2kl'_m)!}{l'_m!}(\lambda^{1/2k}+C)^{2kl'} \tag{32}$$

We deduce a bound of the operation given by (29) in $\frac{t^{l'}C^{2kl'}}{(l')!}(\lambda+C)^{l'}$.

By the same argument, we have a bound of $\frac{t^{l'}}{l'!}(L^k)^{l'}$ on E_λ in $\frac{t^{l'}}{l'!}C^{l'}(\lambda+C)^{l'}$.

In order to conclude, we see that on E_λ

$$Q_t^k = \exp[-\lambda t]Id + \sum_{l'>1}\frac{t^{l'}}{l'!}Q_\lambda^{l',t} \tag{33}$$

where $Q_\lambda^{l',t}$ has a bound on E_λ in $C^{l'}(\lambda+C)^{l'}$. We deduce that Q_t^k acts on E_λ by

$$\exp[-\lambda t]Id + t^2 Q_t^\lambda = R_t^\lambda \tag{34}$$

where the norm on E_λ of Q_t^λ is smaller that $C\exp[C\lambda t]$.

But if $f = \sum a_\lambda f_\lambda$

$$(Q_{1/n}^k)^n f = \sum a_\lambda (R_{1/n}^\lambda)^n f_\lambda \tag{35}$$

Moreover

$$\|(Q_t^k)f\|_{L^2(G)} = \int_G |\int_{\mathbb{R}^m} f(x(t^{1/2k}w)(g)p_1^{f,k}(w)dw|^2 dg \leq$$
$$C\int_G dg \int_{\mathbb{R}^m} |f(x(t^{1/2k}w)(g)|^2 |p_1^{f,k}(w)|^2 dw$$
$$\leq C\int_{\mathbb{R}^m} |p_1^{f,k}(w)|^2 dw \int_G |f(x(t^{1/2k}w)(g)|^2 dg \quad (36)$$

But

$$\int_G |f(x(t^{1/2k}w)(g)|^2 dg = \|f\|^2_{L^2(G)} \quad (37)$$

because the vector fields are without divergence. If $\lambda/n < C$, the sum

$$\sum_{\lambda < Cn} a_\lambda (R^\lambda_{1/n})^n f_\lambda \quad (38)$$

converges to

$$\sum a_\lambda \exp[-\lambda] f_\lambda \quad (39)$$

Moreover $\sum_{\lambda > Cn} a_\lambda (R^\lambda_{1/n})^n f_\lambda$ has a L^2 norm bounded by$(\sum_{\lambda > Cn} | a_\lambda|^2 C^\lambda)^{1/2}$ which goes to 0 when $n \to \infty$.

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